Bound for the pressure integral in a toroidal-plasma equilibrium

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A bound for the pressure integral in a toroidal-plasma equilibrium has been studied, invoking an *a priori* estimate for a solution of the Grad-Shafranov equation. Earlier theories had to use approximate equilibrium solutions to calculate the pressure integral (poloidal beta ratio β_p), and, hence, fell short of being rigorous estimates. The present theory considers exact solutions and gives a rigorous bound for β_p.

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I. INTRODUCTION

In a magnetohydrodynamic equilibrium of a plasma, the thermal pressure force ∇p is balanced by the magnetic stress $\mathbf{j}\times\mathbf{B}$, where $\mathbf{B}$ is the magnetic flux density, $\mathbf{j}=\nabla\times\mathbf{B}/\mu_0$ is the current density in the plasma, and μ_0 is the vacuum permeability. The plasma equilibrium equation $\nabla p=\mathbf{j}\times\mathbf{B}$ thus relates the pressure and the current. In this paper we study the maximum of the poloidal β ratio, which is defined by

$$\beta_p=8\pi\int_\Omega p\,dx/(J^2\mu_0)\ ,$$

where dx is the surface element, the integral is taken over the cross section Ω of the axisymmetric toroidal plasma (tokamak), and J is the toroidal current (total current passing through Ω). Recently high-β_p tokamaks have attracted much interest because they have many advantageous features for a fusion core [1]. Limitation of β_p can occur because of nonexistence of equilibrium solutions, as well as onset of instabilities [2]. The former is called the equilibrium limit, while the latter is the stability limit.

Discussions on the equilibrium limit have a history of confusion. By simplified analytic calculations using approximate equilibrium solutions, one observes a bound $\beta_p\le O(\epsilon^{-1})$, which is determined by formation of a separatrix [3]; see also [4] and [5]. Here $\epsilon=a/R$, R is the major radius, and a is the minor radius of the toroid. The flux-conserving tokamak (FCT) theory, however, predicted that β_p has no bound determined by a separatrix [6]. The FCT concept has been also applied to numerical analyses, and solutions with relatively high β_p have been obtained [7]. When one attempts to increase β_p further, however, a limitation occurs since the convergence of the scheme becomes difficult in the high-β_p regime. When β_p is increased, the flux surfaces are strongly deformed to shift toward the outer edge of the toroid, and the plasma pressure and the current are concentrated into a crescent shape around the outer edge [7]. It has been an open question whether the limitation of the convergence is due to a technical problem of generating meshes or due to the "absence" of solutions. The analytic estimate of the FCT theory [6] assumes circular cross-section flux surfaces, so it requires an appropriate correction to account for strong deformations of flux surfaces in high-β_p solutions. This paper addresses to this critical problem, and derives a rigorous bound for β_p.

The mathematical technique used here is the so-called *a priori* estimate. We do not invoke any approximation or expansion for solutions, while we discuss exact solutions of the basic equation. We construct inequalities which every solution should satisfy in an *a priori* sense.

II. THE GRAD-SHAFRANOV EQUATION

A tokamak equilibrium (an axisymmetric plasma equilibrium) is represented by a flux function Ψ which solves the Grad-Shafranov equation (for example, see [4])

$$A\Psi=rP'(\Psi)+r^{-1}F(\Psi)F'(\Psi)\quad(\text{in }\Omega)\ , \tag{1}$$

$$\Psi=0\quad(\text{on }\partial\Omega)\ , \tag{2}$$

where $A\Psi=-\partial_r(r^{-1}\partial_r\Psi)-r^{-1}\partial_z^2\Psi$ in the r-z coordinates, and the cross section Ω of the toroid is a simply connected bounded domain in $\mathbb{R}^+\times\mathbb{R}$ with a smooth boundary $\partial\Omega$. The boundary condition (2) implies a perfectly conductive wall. We note that $j=A\Psi/\mu_0$ parallels the toroidal current density, $P=\mu_0 p$, and $F=rB_\varphi$, where B_φ is the toroidal magnetic field. In (1), P and F are composite functions of Ψ. To avoid exceedingly mathematical arguments, we consider smooth functions P and F, to warrant smoothness of Ψ. We have denoted $P'(s)=dP(s)/ds$ for $P(s)$: $\mathbb{R}\to\mathbb{R}$. For simplicity, we assume

$$P'\ge 0\ ,\quad P(0)=0\ . \tag{3}$$

Therefore, the pressure peaks at the magnetic axis (the peak of Ψ) and vanishes on $\partial\Omega$. We choose the sign of the current J positive, and assume $\Psi\ge 0$ in Ω.

An essential condition for Ψ to be a permissible solution is that Ψ does not have a separatrix in Ω [4,6]. Formally this condition reads as follows. We denote by $D(s)$

the domain on which $\Psi > s$. The boundary of $D(s)$ is denoted by $L(s)$, which is the level set of $\Psi = s$ $(0 < s < \max\Psi)$. Then, every $L(s)$ should be a simple closed loop in Ω, and

$$-\mathbf{n}\cdot\nabla\Psi = |\nabla\Psi| > 0 \quad \text{on every} L(s) \ , \tag{4}$$

where $\mathbf{n}$ is the outward normal vector on $L(s)$.

III. A BOUND FOR THE PRESSURE INTEGRAL

Before studying the pressure integral in a tokamak equilibrium, we concisely survey related theories. For a circular cross section straight z-pinch plasma column (cylindrical plasma with $\mathbf{j}$ in the longitudinal direction), we have Bennett's pinch relation $\beta_p \equiv 1$, which holds for every profile of the pressure; see, e.g., [5]. For a general shape of cross section, one finds $\beta_p \le 1$. This relation is expectable, since any deformation of the cross section leads to a stretch of the poloidal magnetic-field lines resulting in a decrease in the magnetic stress. Although a rigorous proof of $\beta_p \le 1$ is not found in the literature of plasma physics, the Payne-Rayner inequality [8], which was developed for the fixed membrane problem in solid mechanics, applies to this proof. When a longitudinal magnetic field is imposed on a straight plasma column, a poloidal current yields an additional magnetic stress, and the plasma pressure can be increased infinitely without changing longitudinal current. Therefore, β_p is unbounded in a straight tokamak. A limitation of β_p, however, can arise from the toroidal curvature effect. In this section, we derive a bound for β_p in a toroidal equilibrium with a toroidal (longitudinal) magnetic field applying the method of the Payne-Rayner inequality. A stronger bound results from limiting the rotational transform, which will be discussed in the next section.

Theorem 1. Suppose that P satisfies (3). Let Ψ (≥ 0) be a smooth function in Ω satisfying (2) and (4). Then, one finds

$$\int_\Omega P(\Psi)dx \le \frac{I_m I_p R_1}{4\pi R_0} \ , \tag{5}$$

where $I_m = \max I(s)$,

$$I(s) = \int_{D(s)} A\Psi\, dx \ , \quad I_P = \int_\Omega rP'(\Psi)dx \ ,$$

$R_0 = \min r$ and $R_1 = \max r$ in Ω, respectively.

Proof. The proof is similar to that of the Payne-Rayner inequality [8]. We denote

$$\sigma(s) = \int_{D(s)} dx \ , \quad \vartheta(s) = \int_{D(s)} P'(\Psi)dx \ .$$

We observe

$$\sigma'(s) = -\int_{L(s)} |\nabla\Psi|^{-1} d\Gamma \ ,$$

where $d\Gamma$ is the line element on $L(s)$. Using (4), integrate (1) over $D(s)$ to obtain

$$\int_{D(s)} A\Psi\, ds = \int_{L(s)} r^{-1}|\nabla\Psi| d\Gamma \ge R_1^{-1} \int_{L(s)} |\nabla\Psi| d\Gamma \ . \tag{6}$$

Multiply $-P'\sigma' = -\vartheta'$ [≥ 0 by (3)] on both sides of (6) to obtain

$$-R_1\vartheta'(s) I_m \ge -R_1\vartheta'(s) \int_{D(s)} A\Psi\, dx \ge 4\pi P'(s)\sigma(s) \ . \tag{7}$$

Here we have used the following isoperimetric inequality:

$$\int_{L(s)} |\nabla\Psi| d\Gamma \int_{L(s)} |\nabla\Psi|^{-1} d\Gamma \ge \left[\int_{L(s)} d\Gamma\right]^2 \ge 4\pi\sigma(s) \ .$$

Integrating (7) with respect to s over $(0, \max\Psi)$ yields

$$4\pi \int_\Omega P\, dx \le R_1 I_m \vartheta(0) \le \frac{R_1}{R_0} I_m I_p \ . \qquad \text{Q.E.D.}$$

Estimate (5) reads as $\beta_p \le 2(R_1 I_m I_p)/(R_0 I^2)$, where $I = I(0) = \mu_0 J$. From now on we assume that Ψ solves (1) and (2). If the force-free current $r^{-1}FF'/\mu_0$ is positive, $I_p < I$, and $I_m < I$ by (3), and hence $\beta_p \le 2R_1/R_0$. To achieve a larger β_p, $r^{-1}FF'/\mu_0$ should be allowed negative. In a high-β_p equilibrium, the toroidal diamagnetism enhances the pressure, and hence $r^{-1}FF'/\mu_0$ tends to be negative. Because $|r^{-1}FF'|$ is large on the inner side, while rP' is large on the outer side, a negative current region can develop on the inner side of the toroid. This is known as the Pfirsh-Schlüter current (for example, see [5]).

IV. FCT EQUILIBRIA

In what follows we derive a bound for the negative current considering an additional restriction on the safety factor $q(\Psi)$. This restriction on q is relevant to the FCT set of equilibria [6,7]. The safety factor is given by

$$q(s) = \frac{F(s)}{2\pi} \int_{L(s)} r^{-1}|\nabla\Psi|^{-1} d\Gamma \ . \tag{8}$$

We consider a set of equilibria such that

$$0 < q(s) \le q_m \ , \quad 0 \le s \le \max\Psi \ . \tag{9}$$

To simplify estimates, we also assume that $F'(s)$ does not change the sign. Since $q > 0$, we observe $F > 0$, so we should assume, for high β_p,

$$F' \le 0 \ . \tag{10}$$

Theorem 2. Suppose that P satisfies (3), and that F satisfies (10). Let Ψ be a smooth solution of (1) and (2), which satisfies (4) and (9). Then, one finds an *a priori* bound for β_p:

$$\beta_p(\Psi) \le 2\frac{R_1}{R_0}\left[1 + \frac{q_m^2 R_1^3}{2\sigma(0)R_0}\right]^2 \ . \tag{11}$$

Proof. By (8) and (9), we observe

$$-F(s)\sigma'(s) \le F(s) R_1 \int_{L(s)} r^{-1}|\nabla\Psi|^{-1} d\Gamma \le 2\pi R_1 q_m \ .$$

Using this relation and (10), we obtain

$$\begin{aligned} -I_f := -\int_\Omega r^{-1}F(\Psi)F'(\Psi)dx &\le R_0^{-1} \int F(s)F'(s)\sigma'(s)ds \\ &\le -2\pi q_m \frac{R_1}{R_0} \int F'(s)ds \le 2\pi q_m \frac{R_1}{R_0} F(0) \ . \end{aligned} \tag{12}$$

The isoperimetric inequality yields

$$4\pi\sigma(0) \le R_1^2 \int_{L(0)} r^{-1}|\nabla\Psi| d\Gamma \int_{L(0)} r^{-1}|\nabla\Psi|^{-1} d\Gamma$$
$$= R_1^2 I \int_{L(0)} r^{-1}|\nabla\Psi|^{-1} d\Gamma \ . \quad (13)$$

Using (8) and (13), we obtain

$$F(0) \le 2\pi q_m \Big/ \int_{L(0)} r^{-1}|\nabla\Psi|^{-1} d\Gamma \le \frac{q_m R_1^2 I}{2\sigma(0)} \ . \quad (14)$$

Combine (5), (12), (14), $I_p = I - I_f$, and $I_m \le I - I_f$ to obtain (11) Q.E.D.

V. SUMMARY AND DISCUSSION

In summary we have derived rigorous bounds for the pressure integral in an axisymmetric plasma equilibrium. Theorem 1 is an extension of the Payne-Rayner inequality to the toroidal problem with the additional force-free current term $r^{-1}FF'$. Theorem 2 gives an explicit bound for a specific set of FCT equilibria showing that β_p is bounded by a number that is a function of q_m and the geometry. We note that $\sigma(0)$ represents the area of Ω. Our estimate (11) may be improved by excluding extraordinary configurations. For example, when we assume that the flux-surface average of the current density should be positive, then $I_m = I$, and hence we have

$$\beta_p(\Psi) \le 2\frac{R_1}{R_0}\left[1 + \frac{q_m^2 R_1^3}{2\sigma(0)R_0}\right] .$$

On the other hand, if we allow F' to change the sign n times ($n \ge 1$), we should modify (12) to a complicated form. A crude estimate is given by multiplying by n the bound of (11). This pushes the bound up, while it is unlikely to have a large n.

Another important question is the relation between β_p and the shape of an equilibrium solution. Previous theories used an asymptotic parameter ϵ (inverse aspect ratio), indicating that $\epsilon\beta_p \le O(1)$ even if q is unbounded. The FCT model of Clarke and Sigmar [6] estimates $\epsilon\beta_p \sim (\beta/\epsilon)^{1/3}$, where $\beta = 2\mu_0 \int_\Omega p\, dx \,/ \int_\Omega B^2 dx$. In these models, an equilibrium is approximated by a circular level set Ψ, and ϵ is defined by the outermost magnetic surface. As is well known by numerical analyses, a high-β_p equilibrium has a narrow confinement region localized at the outer edge of the toroid. Therefore the definition of ϵ becomes difficult for general high-β_p equilibria. One might expect that β_p is bounded by a number that is a function of only the shape of the boundary $\partial\Omega$, instead of the solution. Cowley *et al.* [9] used asymptotic methods assuming boundary-layer types of equilibria instead of circular level-set equilibria, and interesting estimates of the pressure integral were obtained. Our general result (11) is weaker than those heuristic arguments, while it is on a rigorous mathematical basis. We can define an appropriate scale length of the confinement region (not the boundary of the domain), and define an effective aspect ratio $1/\epsilon^*$ of the solution. Then we obtain an estimate $\epsilon^*\beta_p \le 2$. Such detailed analyses will be discussed elsewhere.

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